

## Estimation of Trends in the Presence of Measurement Error

Definition 1 Consider the covariance stationary stochastic process  $\{\epsilon_t, \eta_t, x_t\}_{t=0}^{\infty}$ . Let the correlation between  $x_t$  and  $x_{t-1}$  be  $r < 1$ .

Definition 2 Let the correlation between  $t$  and  $x_t$  be the constant  $\rho \in (-1, 1)$ .

Definition 3  $y_t \equiv \alpha + \gamma t + \beta x_t + \epsilon_t$

Definition 4  $x_t^* \equiv x_t + \eta_t$

Definition 5 From a sample  $t=0 \dots T$ , let  $\hat{\beta}_{lev}$  be the coefficient on  $x_t$  from a least squares regression of  $y_t$  on  $t$ ,  $x_t^*$  and a constant.

Definition 5 From a sample  $t=0 \dots T$ , let  $\hat{\beta}_{diff}$  be the coefficient on  $(x_t^* - x_{t-1}^*)$  from a least squares regression of  $(y_t - y_{t-1})$  on  $(x_t^* - x_{t-1}^*)$  and a constant.

Assumption 1 Let the following covariances be zero:

$$\begin{aligned} c(x_t, \epsilon_t) &= c(x_t, \eta_t) = c(x_t, \epsilon_{t-1}) = c(x_{t-1}, \epsilon_t) = c(x_t, \eta_{t-1}) = \\ c(x_{t-1}, \eta_t) &= c(\epsilon_{t-1}, \eta_t) = c(\epsilon_t, \eta_{t-1}) = c(\epsilon_t, \epsilon_{t-1}) = c(\eta_t, \eta_{t-1}) = 0 \end{aligned}$$

Assumption 2  $v(\eta) > 0$

Assumption 3  $\beta > 0$

Assumption 4  $c(\epsilon_t, \eta_t) \leq 0$

Assumption 5  $r > \rho^2$

Lemma 1

$$plim \hat{\beta}_{lev} = \beta \frac{v(x)(1 - \rho^2)}{v(x)(1 - \rho^2) + v(\eta)} + \frac{c(\epsilon, \eta)}{v(x)(1 - \rho^2) + v(\eta)}$$

Proof (i) Substitute definitions 1-4 into the formula for the probability limit of a multiple regression coefficient.  
 (ii) Use  $c(x_t, \epsilon_t) = c(x_t, \eta_t) = 0$

Lemma 2

$$plim \hat{\beta}_{diff} = \beta \frac{v(x)(1-r)}{v(x)(1-r) + v(\eta)} + \frac{\alpha(\varepsilon, \eta)}{v(x)(1-r) + v(\eta)}$$

- Proof
- (i) Substitute definitions 1-4 into the formula for the probability limit of a simple regression coefficient.
  - (ii) Use Assumption 1.

Proposition 1

$$plim \hat{\beta}_{lev} < \beta$$

Proof Use Lemma 1 and Assumptions 2-4.

Proposition 2

$$plim \hat{\beta}_{diff} < \beta$$

Proof Use Lemma 2 and Assumptions 2-4.

Proposition 3

$$plim \hat{\beta}_{diff} < plim \hat{\beta}_{lev}$$

Proof Subtract the expression in Lemma 2 from that in Lemma 1. Assumption 5 delivers the inequality above.